PERGAMON

International Journal of Solids and Structures 40 (2003) 6937–6955

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Nonuniform transformation field analysis

J.C. Michel, P. Suquet *

Laboratoire de Mécanique et d'Acoustique, Centre National de la Recherche Scientifique, 31 Chemin Joseph Aiguier, 13402 Marseille Cedex 20, France

Received 8 October 2002

Abstract

The exact description of the overall behavior of composites with nonlinear dissipative phases requires an infinity of internal variables. Approximate models involving only a finite number of those can be obtained by considering a decomposition of the microscopic anelastic strain field on a finite set of transformation fields. The *Transformation Field Analysis* of Dvorak [Proc. R. Soc. Lond. A 437 (1992) 311] corresponds to piecewise uniform transformation fields. The present theory considers *nonuniform* transformation fields. Comparison with numerical simulations shows the accuracy of the proposed model.

Keywords: Composite materials; Plasticity; Viscoplasticity; Transformation strain analysis

1. Introduction

The present study is devoted to the prediction of the effective mechanical behavior of composite materials comprised of individual constituents which are "standard" materials in a generalized sense, the most common standard materials being elastoviscoplastic or elastoplastic.

The theory of *Generalized Standard Materials* (GSM), proposed in the seminal work of Halphen and Nguyen (1975), is an elegant and powerful generalization of the classical theory of (visco)plasticity. It has been successfully applied and extended to complex material behaviors (Germain et al., 1983; Lemaitre and Chaboche, 1988; Lubliner, 1990; Maugin, 1992). The theory relies on two fundamental concepts, the notion of internal variables, and the notion of thermodynamic potentials, the free energy w and the dissipation potential φ, endowed with specific mathematical properties (convexity). The internal variables α at time t are supposed to contain all the relevant information about the material history for times $\tau \leqslant t$. The choice of these variables depends obviously on the material under consideration.

It has been known for some time that the structure of GSM is preserved under change of scales. In other words the effective behavior of composites made of constituents which are GSM, has itself a GSM structure but with infinitely many internal variables which are the fields of local internal variables. This fairly

* Corresponding author. Tel.: +33-4-9116-4005; fax: +33-4-9171-2866.
E-mail address: suquet@lma.cnrs-mrs.fr (P. Suquet).

0020-7683/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0020-7683(03)00346-9

theoretical result is recalled in Section 2 following the presentation of Suquet (1985), but similar ideas were already present in different forms in the works of Bui (1970), Rice (1970) and Mandel (1972) among others.

This general result shows the complexity of nonlinear homogenization, but is of very limited practical use. In order to derive constitutive models of the effective behavior of composites which are both useable and reasonably accurate, one has to resort either to bounding (variational) principles or to other types of approximation.

Regarding variational methods, significant progress has been made recently in deriving bounds and estimates for nonlinear composites from variational principles (Talbot and Willis, 1985; Ponte Castañeda, 1991, 1992, 1996; Willis, 1991; Suquet, 1992, 1993; Olson, 1994; Ponte Castañeda and Suquet, 1997). However, most of these studies apply only to nonlinear constitutive behaviors deriving from a single potential, such as nonlinear elasticity or viscoplasticity when elastic effects are neglected. The question of nonlinear constitutive behaviors governed by two potentials corresponding respectively to reversible and irreversible aspects of the behavior is still widely open.

Regarding approximate schemes, the *Transformation Field Analysis* (TFA) proposed by Dvorak and coworkers, initially for elastoplastic composites (Dvorak and Rao, 1976; Dvorak and Bahei-El-Din, 1987; Dvorak et al., 1988; Teply and Dvorak, 1988), is an elegant way of reducing the number of macroscopic internal variables by assuming the *microscopic* fields of internal variables to be piecewise uniform. Its formalization by Dvorak and Benveniste (1992) and Dvorak (1992) has provided a theoretical basis for further work with more complex behavior (thermoviscoplasticity, damage) by Dvorak et al. (1994) and by other groups (Kattan and Voyiadjis, 1993; Chaboche et al., 2001; Fish et al., 1997; Fish and Yu, 2002), this list being by no means exhaustive.

The TFA has been used by Fish et al. (1997) to analyse a composite structure by the FEM. These authors found a good agreement between the "two-point averaging scheme" (plain TFA with a uniform plastic strain in the matrix) and a more refined computation ("multi-point incremental homogenization"). Despite this interesting observation, it has long been recognized by Dvorak himself (Teply and Dvorak, 1988) and confirmed by others (Suquet, 1997; Chaboche et al., 2001; Michel et al., 2000) that the application of the TFA to two-phase systems may require, under certain circumstances, a subdivision of each individual phase into several (and sometimes numerous) sub-domains to obtain a satisfactory description of the effective behavior. As a consequence, the number of internal variables needed in the effective constitutive relations, although finite, is prohibitively high. The need for a finer subdivision of the phases stems from the intrinsic nonuniformity of the plastic strain field which can be highly heterogeneous even within a single material phase. In order to reproduce accurately the actual effective behavior of the composite, it is important to capture correctly the heterogeneity of the plastic strain field.

This observation has motivated the method proposed in a previous study (Michel et al., 2000) and in the present study with the aim of reducing the number of macroscopic internal variables by considering *non-uniform plastic strain fields*. More specifically the plastic strain within each phase is decomposed on a finite set of *plastic modes* which can present large deviations from uniformity. The plastic modes considered in Michel et al. (2000) were scalar modes. The present study is an extension of the theory to tensorial plastic modes. The method is exposed in Section 4. It is illustrated in Section 5 where its merits are assessed by comparison with the TFA and with complete numerical simulations.

2. Constitutive relations defined by two thermodynamic potentials

2.1. Generalized standard materials

It is assumed that the constituents of the composite are GSM in the sense of Halphen and Nguyen (1975) or Germain et al. (1983). At each material point x the state variables are the (infinitesimal) deformation

tensor ε and additional internal variables $\boldsymbol{\alpha}$ which describe irreversible phenomena (plasticity, viscoplasticity, damage,...). The evolution equations governing the behavior of the material are derived from two thermodynamic potentials. The free energy w is a convex functions of its arguments which defines (through the state laws) the stress $\boldsymbol{\sigma}$ and the forces $\mathscr{A}$ available in the system to drive the dissipative mechanisms

$$\text{State laws}: \quad \boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}), \quad \mathscr{A} = -\frac{\partial w}{\partial \boldsymbol{\alpha}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}). \tag{1}$$

The rate of the internal variables and the associated forces are related by means of the *dissipation potential* φ or equivalently by means of the *force potential* ψ (dual to φ) through the *complementary laws*

$$\text{Complementary laws}: \quad \mathscr{A} = \frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}}(\dot{\boldsymbol{\alpha}}) \quad \text{or equivalently} \quad \dot{\boldsymbol{\alpha}} = \frac{\partial \psi}{\partial \mathscr{A}}(\mathscr{A}). \tag{2}$$

ψ and φ are dual *convex* potentials.

2.2. A sub-class of generalized standard materials (GSM2)

With the classical theories of elastoplasticity and elastoviscoplasticity in mind, attention can be restricted to an important sub-class of GSM. The internal variables consist of the anelastic strain $\boldsymbol{\varepsilon}^{\mathrm{an}}$ and additional state variables $\boldsymbol{\beta}$. The free energy is split into three terms, a purely elastic energy not affected by the variables $\boldsymbol{\beta}$ and a stored energy consisting of two terms, the energy stored in the kinematic hardening of the material and the energy stored in other mechanisms and described by the variables $\boldsymbol{\beta}$

$$\boldsymbol{\alpha} = (\boldsymbol{\varepsilon}^{\mathrm{an}}, \beta), \quad w(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}) : \boldsymbol{L} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}) + w^{\mathrm{an}}(\boldsymbol{\varepsilon}^{\mathrm{an}}) + w^{\boldsymbol{\beta}}(\boldsymbol{\beta}).$$

The associated forces are

$$\left.\begin{aligned} &\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \boldsymbol{L} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}), \\ &\mathscr{A}^{\mathrm{an}} = -\frac{\partial w}{\partial \boldsymbol{\varepsilon}^{\mathrm{an}}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \boldsymbol{\sigma} - \boldsymbol{X}, \quad \boldsymbol{X} = \frac{\partial w^{\mathrm{an}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{an}}}(\boldsymbol{\varepsilon}^{\mathrm{an}}), \\ &\mathscr{A}^{\boldsymbol{\beta}} = -\frac{\partial w}{\partial \boldsymbol{\beta}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = -\frac{\partial w^{\boldsymbol{\beta}}}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}). \end{aligned}\right\} \tag{3}$$

$\boldsymbol{X}$ is usually called the back stress.

The constitutive relations (3) can be further simplified when the material under consideration is isotropic. When this is the case:

(a) The elastic moduli $\boldsymbol{L}$ can be expressed in terms of two materials parameters, the bulk modulus k and the shear modulus G.

(b) The energy function w^{an} is a function of $\varepsilon^{\mathrm{an}}_{\mathrm{eq}}$

$$w^{\mathrm{an}}(\boldsymbol{\varepsilon}^{\mathrm{an}}) = w^{\mathrm{an}}(e) \quad e = \varepsilon^{\mathrm{an}}_{\mathrm{eq}} = \left(\frac{2}{3}\boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\varepsilon}^{\mathrm{an}}\right)^{1/2}. \tag{4}$$

(c) The force potential is a function of $\mathscr{A}^{\mathrm{an}}_{\mathrm{eq}}$ and $\mathscr{A}^{\boldsymbol{\beta}}$

$$\psi(\mathscr{A}^{\mathrm{an}}, \mathscr{A}^{\boldsymbol{\beta}}) = \psi(a, \mathscr{A}^{\boldsymbol{\beta}}), \quad a = \mathscr{A}^{\mathrm{an}}_{\mathrm{eq}} = \left(\frac{3}{2}\mathscr{A}^{\mathrm{an}}_{\mathrm{dev}} : \mathscr{A}^{\mathrm{an}}_{\mathrm{dev}}\right)^{1/2}, \tag{5}$$

where $\mathscr{A}^{\mathrm{an}}_{\mathrm{dev}}$ is the deviator of $\mathscr{A}^{\mathrm{an}}$. The forms (4) and (5) of the potentials lead to significant simplifications for the constitutive relations

$$\boldsymbol{X} = \frac{2}{3}\frac{\partial w^{\mathrm{an}}}{\partial e}(\varepsilon^{\mathrm{an}}_{\mathrm{eq}})\frac{\boldsymbol{\varepsilon}^{\mathrm{an}}}{\varepsilon^{\mathrm{an}}_{\mathrm{eq}}}, \quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{an}} = \frac{3}{2}\frac{\partial \psi}{\partial a}(\mathscr{A}^{\mathrm{an}}_{\mathrm{eq}}, \mathscr{A}^{\boldsymbol{\beta}})\frac{\mathscr{A}^{\mathrm{an}}_{\mathrm{dev}}}{\mathscr{A}^{\mathrm{an}}_{\mathrm{eq}}}. \tag{6}$$

Example: *plasticity and viscoplasticity*. Several examples of constitutive relations in the form (1) and (2) can be found in Germain et al. (1983), Lemaitre and Chaboche (1988), Maugin (1992). The classical example of viscoplasticity (or plasticity) with kinematic and isotropic hardening fits in the class GSM2 and is worth mentioning explicitely. For this class of materials, the internal variables are the anelastic (plastic or viscoplastic) strain and a scalar variable modelling the effect of isotropic hardening

$$\boldsymbol{\alpha} = (\boldsymbol{\varepsilon}^{\mathrm{an}}, p).$$

The free energy w reads as

$$w(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{an}}, p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}) : \boldsymbol{L} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}) + \frac{1}{2} H \boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\varepsilon}^{\mathrm{an}} + w^{\mathrm{p}}(p), \tag{7}$$

where $\mathbf{L}$ is the fourth-order tensor of elastic moduli, H is the kinematic hardening modulus, $w^{\mathrm{an}} = (1/2)H\boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\varepsilon}^{\mathrm{an}}$ and w^{p} are energies which are stored in the kinematic and isotropic hardening of the material. The state laws (1) defining the stress and the forces associated to the internal variables read in this case

$$\left.\begin{aligned} &\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \boldsymbol{L} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{an}}), \\ &\mathscr{A}^{\mathrm{an}} = -\frac{\partial w}{\partial \boldsymbol{\varepsilon}^{\mathrm{an}}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \boldsymbol{\sigma} - \boldsymbol{X}, \quad \boldsymbol{X} = H\boldsymbol{\varepsilon}^{\mathrm{an}}, \\ &\mathscr{A}^{\mathrm{p}} = -\frac{\partial w}{\partial p}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = -\frac{\partial w^{\mathrm{p}}}{\partial p}(p) = -R(p). \end{aligned}\right\} \tag{8}$$

In classical viscoplasticity the force potential reads as

$$\psi(\mathscr{A}^{\mathrm{an}}, \mathscr{A}^{\mathrm{p}}) = \frac{\sigma_0 \dot{\varepsilon}_0}{n+1}\left[\frac{(\mathscr{A}^{\mathrm{an}}_{\mathrm{eq}} + \mathscr{A}^{\mathrm{p}})^+}{\sigma_0}\right]^{n+1} = \frac{\sigma_0 \dot{\varepsilon}_0}{n+1}\left[\frac{((\sigma - X)_{\mathrm{eq}} - R(p))^+}{\sigma_0}\right]^{n+1}, \tag{9}$$

and the complementary laws are

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{an}} = \frac{3}{2}\dot{p}\frac{\boldsymbol{s} - \boldsymbol{X}}{(\sigma - X)_{\mathrm{eq}}}, \quad \dot{p} = \dot{\varepsilon}_0\left[\frac{((\sigma - X)_{\mathrm{eq}} - R(p))^+}{\sigma_0}\right]^n,$$

where $\boldsymbol{s}$ is the stress deviator.

2.3. Local problem for composite materials

2.3.1. Notations

Consider a *representative volume element* (r.v.e) V of a composite material comprised of N homogeneous phases or subdomains V_r. By subdomain it is understood that a single mechanical phase can be subdivided into several subdomains considered as different phases, although their material properties are identical. $\chi^{(r)}(\boldsymbol{x})$ and c_r denote the characteristic function and the volume fraction of phase r

$$\chi^{(r)}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in V_r, \\ 0 & \text{otherwise,} \end{cases} \quad c_r = \langle \chi^{(r)} \rangle \quad \text{where } \langle f \rangle = \frac{1}{|V|}\int_V f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The compact notations $\bar{f}$ and $\bar{f}_r$ will be used to denote the average of a field f in the entire r.v.e. V and in each phase V_r, respectively

$$\bar{f} = \langle f \rangle = \sum_{r=1}^{N} c_r \bar{f}_r, \quad \bar{f}_r = \langle f \rangle_r = \frac{1}{|V_r|}\int_{V_r} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The r.v.e is subjected to an average loading characterized by a given path in the space of overall strain or stress. Attention is limited to isothermal evolutions and infinitesimal deformations. The overall stress $\bar{\boldsymbol{\sigma}}$ and the overall strain $\bar{\boldsymbol{\varepsilon}}$ are the averages of their local counterparts $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$

$$\bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle, \quad \bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle. \tag{10}$$

The local stress and strain fields are determined through the resolution of the *local* evolution problem posed for the r.v.e. and consisting of equilibrium equations, boundary conditions and constitutive relations. The boundary conditions are assumed to be such that Hill's micro–macro localization condition is satisfied: for any compatible strain field $\boldsymbol{\varepsilon}$ and any stress field $\boldsymbol{\sigma}$ in equilibrium, both meeting the boundary conditions imposed on the boundary of the r.v.e., the following equality holds:

$$\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle = \langle \boldsymbol{\sigma} \rangle : \langle \boldsymbol{\varepsilon} \rangle. \tag{11}$$

Examples of boundary conditions meeting (11) include uniform strains, uniform stresses, periodicity conditions (see Suquet (1987) for more details). Periodicity boundary conditions will be assumed in the following.

2.4. Generalized structure of the overall constitutive relations for composite materials

As shown in Suquet (1985, 1987), the "standard" structure of the constitutive relations (1) and (2) is preserved by change of scales at the expense of introducing an *infinite number of internal variables*. Let us briefly describe how this (highly theoretical) result is obtained.

The internal variables at the macroscopic scale are the fields of internal variables at each microscopic location $\boldsymbol{x} \in V$

$$\tilde{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha}(\boldsymbol{x})\}_{\boldsymbol{x}\in V}. \tag{12}$$

Since the free energy is an additive quantity, the overall free energy of the composite is the average of the microscopic free energy

$$\tilde{w}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}) = \langle w(\boldsymbol{\varepsilon}(\boldsymbol{x}), \boldsymbol{\alpha}(\boldsymbol{x})) \rangle. \tag{13}$$

The forces associated with the state variables read

$$\frac{\partial \tilde{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}), \quad \tilde{\mathscr{A}} = \{\mathscr{A}_{\boldsymbol{x}}\}_{\boldsymbol{x}\in V}, \quad \mathscr{A}_{\boldsymbol{x}} = -\frac{\partial \tilde{w}}{\partial \boldsymbol{\alpha}(x)}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}). \tag{14}$$

The first force is nothing else than the macroscopic stress. Indeed, using Hill's lemma, one gets that

$$\frac{\partial \tilde{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}) = \left\langle \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) : \frac{\partial \boldsymbol{\varepsilon}}{\partial \bar{\boldsymbol{\varepsilon}}} \right\rangle = \left\langle \boldsymbol{\sigma} : \frac{\partial \boldsymbol{\varepsilon}}{\partial \bar{\boldsymbol{\varepsilon}}} \right\rangle = \langle \boldsymbol{\sigma} \rangle : \left\langle \frac{\partial \boldsymbol{\varepsilon}}{\partial \bar{\boldsymbol{\varepsilon}}} \right\rangle = \bar{\boldsymbol{\sigma}},$$

since

$$\langle \boldsymbol{\varepsilon} \rangle = \bar{\boldsymbol{\varepsilon}} \quad \text{and thus} \quad \left\langle \frac{\partial \boldsymbol{\varepsilon}}{\partial \bar{\boldsymbol{\varepsilon}}} \right\rangle = \boldsymbol{I}.$$

Similarly, the set of forces $\{\mathscr{A}_{\boldsymbol{x}}\}_{\boldsymbol{x}\in V}$ coincide with the local field of forces $\{\mathscr{A}(\boldsymbol{x})\}_{\boldsymbol{x}\in V}$. The effective force potential reads

$$\tilde{\psi}(\tilde{\mathscr{A}}) = \langle \psi(\mathscr{A}) \rangle. \tag{15}$$

With the choices (12), (13), (15), the effective constitutive relations of the composite have a generalized standard structure

$$\begin{aligned}
&\text{State variables :} \quad \bar{\boldsymbol{\varepsilon}},\ \tilde{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha}(\boldsymbol{x})\}_{\boldsymbol{x}\in V},\\
&\text{State laws :} \quad \bar{\boldsymbol{\sigma}} = \frac{\partial \tilde{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}), \quad \tilde{\mathscr{A}} = -\frac{\partial \tilde{w}}{\partial \tilde{\boldsymbol{\alpha}}}(\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}),\\
&\text{Complementary laws :} \quad \dot{\tilde{\boldsymbol{\alpha}}} = \frac{\partial \tilde{\psi}}{\partial \tilde{\mathscr{A}}}(\tilde{\mathscr{A}}).
\end{aligned} \tag{16}$$

The interest of this result is essentially theoretical since it shows that the structure of generalized standard materials is preserved under change of scale. But the result itself is hardly applicable since the number of internal variables is infinite. The aim of the next sections is to explore different models approximating the exact model but involving only a finite number of internal variables.

2.5. Green's operator $\boldsymbol{\Gamma}$

When the state variables are frozen (no evolution of the system), the stress and strain fields in the r.v.e. solve the following linear elastic problem, with appropriate boundary conditions:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x}) : (\boldsymbol{\varepsilon}(\boldsymbol{x}) - \boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x})), \quad \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{x})) = 0, \quad \langle \boldsymbol{\varepsilon} \rangle = \bar{\boldsymbol{\varepsilon}}. \tag{17}$$

$\bar{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x})$ being known, the solution $\boldsymbol{\varepsilon}$ of this problem can be obtained by a mere application of the superposition principle. Consider first the case where $\boldsymbol{\varepsilon}^{\mathrm{an}}$ is identically $\mathbf{0}$. Problem (17) is then a standard elasticity problem and its solution can be expressed by means of the the elastic strain-localization tensor $\boldsymbol{A}(\boldsymbol{x})$ as

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) : \bar{\boldsymbol{\varepsilon}}. \tag{18}$$

Consider next the case where $\bar{\boldsymbol{\varepsilon}} = 0$ and $\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x})$ is arbitrary. Problem (17) can then be written as an elasticity problem with eigenstress $\boldsymbol{\tau}$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{x}), \quad \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{x})) = 0, \quad \langle \boldsymbol{\varepsilon} \rangle = \mathbf{0}, \tag{19}$$

where $\boldsymbol{\tau}(\boldsymbol{x}) = -\boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x})$. The solution of (19) can be expressed by means of the nonlocal elastic Green operator $\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{x}')$ of the nonhomogeneous elastic medium as

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = -\langle \boldsymbol{\Gamma} * \boldsymbol{\tau} \rangle(\boldsymbol{x}), \quad \text{where } \boldsymbol{\Gamma} * \boldsymbol{\tau}(\boldsymbol{x}) \stackrel{\text{def}}{=} \frac{1}{|V|} \int_V \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{x}') : \boldsymbol{\tau}(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}'. \tag{20}$$

It follows from the superposition principle applied to (18) and (20) that the solution of (17) reads as

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) : \bar{\boldsymbol{\varepsilon}} + \frac{1}{|V|} \int_V \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{x}') : \boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}' = \boldsymbol{A}(\boldsymbol{x}) : \bar{\boldsymbol{\varepsilon}} + \boldsymbol{D} * \boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x}), \tag{21}$$

where the nonlocal operator $\boldsymbol{D}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{x}') : \boldsymbol{L}(\boldsymbol{x}')$ gives the strain at point $\boldsymbol{x}$ created by a transformation strain $\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x}')$ at point $\boldsymbol{x}'$.

3. The classical transformation field analysis (TFA)

The "Transformation Field Analysis" (TFA) of Dvorak (1992) can be used to reduce the number of internal variables. Although initially proposed in a slightly different spirit, the TFA can be adapted to the above described class of generalized standard materials (GSM2). The "transformation fields" are, in the present case, the fields of internal variables $\boldsymbol{\alpha}$.

3.1. Uniform transformation fields

This application involves two steps and two levels of approximation.

H1. The internal variables $\boldsymbol{\alpha}$, including the anelastic strain, are assumed to be piecewise uniform within each individual phase or subdomain:

$$\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x}) = \sum_{r=1}^{N} \boldsymbol{\varepsilon}_r^{\mathrm{an}} \chi^{(r)}(\boldsymbol{x}), \quad \boldsymbol{\beta}(\boldsymbol{x}) = \sum_{r=1}^{N} \boldsymbol{\beta}_r \chi^{(r)}(\boldsymbol{x}). \tag{22}$$

The $\boldsymbol{\alpha}_r = (\boldsymbol{\varepsilon}_r^{\mathrm{an}}, \boldsymbol{\beta}_r)$ are the new internal variables (and there is only a finite number of them). These variables being fixed, one can make use of the equilibrium equations and state equations to compute the average strain in each subdomain. Usually the problem to be solved amounts to a problem with transformation strains for a N-phase composite material. This problem can be solved (at least theoretically) and the result can be expressed as a linear relation between the internal variables $\boldsymbol{\alpha}_r$'s and the average strain (see Section 3.2 for more details)

$$\bar{\boldsymbol{\varepsilon}}_r = \langle \boldsymbol{\varepsilon} \rangle_r = \boldsymbol{A}_r : \bar{\boldsymbol{\varepsilon}} + \sum_{s=1}^{N} \boldsymbol{D}_{rs} : \boldsymbol{\varepsilon}_s^{\mathrm{an}}, \quad r = 1, \ldots, N, \tag{23}$$

where $\boldsymbol{A}_r$ and $\boldsymbol{D}_{rs}$ are tensors depending on the *linear elastic* properties of the individual phases (see Section 3.2 for details). Then, since the stress and the elastic strain are linearly related, the average stress in each individual phase and in the composite can be computed

$$\bar{\boldsymbol{\sigma}}_r = \langle \boldsymbol{\sigma} \rangle_r = \boldsymbol{L}^{(r)} : (\bar{\boldsymbol{\varepsilon}}_r - \boldsymbol{\varepsilon}_r^{\mathrm{an}}), \quad \bar{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c^{(r)} \bar{\boldsymbol{\sigma}}_r. \tag{24}$$

It remains to specify the evolution of the internal variables $\boldsymbol{\alpha}_r = (\boldsymbol{\varepsilon}_r^{\mathrm{an}}, \boldsymbol{\beta}_r)$. These evolution equations *cannot* be deduced from the exact evolution equations (2). A second level of approximation is required.

H2. The evolution of the internal variables $\boldsymbol{\alpha}_r$ follows exactly the constitutive relations in phase r the strain being estimated as the average strain in phase r. In other words, the forces $\mathscr{A}_r$ being defined as

$$\mathscr{A}_r = -\frac{\partial w^{(r)}}{\partial \boldsymbol{\alpha}} (\bar{\boldsymbol{\varepsilon}}_r, \boldsymbol{\alpha}_r). \tag{25}$$

The evolution equations for the internal variables are

$$\dot{\boldsymbol{\alpha}}_r = \frac{\partial \psi^{(r)}}{\partial \mathscr{A}} (\mathscr{A}_r). \tag{26}$$

3.2. Influence tensors $\boldsymbol{D}_{rs}$

Using the decomposition (22) into (21), the average strains $\bar{\boldsymbol{\varepsilon}}_r$ in the different subdomains are found as

$$\bar{\boldsymbol{\varepsilon}}_r = \boldsymbol{A}_r : \bar{\boldsymbol{\varepsilon}} + \sum_{s=1}^{N} \boldsymbol{D}_{rs} : \boldsymbol{\varepsilon}_s^{\mathrm{an}}, \quad r = 1, \ldots, N, \tag{27}$$

where the fourth-order tensors $\boldsymbol{A}_r$ and $\boldsymbol{D}_{rs}$ are the average strain-localization tensors and influence tensors (Dvorak (1992))

$$\boldsymbol{A}_r = \frac{1}{c_r}\frac{1}{|V|}\int_V \boldsymbol{A}(\boldsymbol{x})\chi_r(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and

$$\boldsymbol{D}_{rs} = \frac{1}{c_r}\frac{1}{|V|}\frac{1}{|V|}\int_V\int_V \chi_r(\boldsymbol{x})\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{x}') : \boldsymbol{L}(\boldsymbol{x}')\chi_s(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}.$$

This completes the proof of (23).

The average stress in phase r is

$$\bar{\boldsymbol{\sigma}}_r = \boldsymbol{L}^{(r)} : \boldsymbol{A}_r : \bar{\boldsymbol{\varepsilon}} + \boldsymbol{L}^{(r)} : \left(\sum_{s=1}^{N} \boldsymbol{D}_{rs} : \boldsymbol{\varepsilon}_s^{\mathrm{an}} - \boldsymbol{\varepsilon}_r^{\mathrm{an}}\right), \quad r = 1,\ldots,N, \tag{28}$$

and the total average stress reads

$$\bar{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c_r\bar{\boldsymbol{\sigma}}_r = \tilde{\boldsymbol{L}} : \bar{\boldsymbol{\varepsilon}} + \sum_{r=1}^{N} c_r\boldsymbol{L}^{(r)} : \sum_{s=1}^{N}(\boldsymbol{D}_{rs} - \delta_{rs}) : \boldsymbol{\varepsilon}_s^{\mathrm{an}}, \tag{29}$$

with

$$\tilde{\boldsymbol{L}} = \sum_{r=1}^{N} c_r\boldsymbol{L}^{(r)} : \boldsymbol{A}_r.$$

In summary the system of equations defining the effective behavior of the composite consists of (29) complemented with evolution equation for the internal variables

$$\dot{\boldsymbol{\varepsilon}}_r^{\mathrm{an}} = \frac{\partial\psi^{(r)}}{\partial\mathscr{A}^{\mathrm{an}}}(\mathscr{A}_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}}), \quad \dot{\boldsymbol{\beta}}_r = \frac{\partial\psi^{(r)}}{\partial\mathscr{A}^{\boldsymbol{\beta}}}(\mathscr{A}_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}}), \tag{30}$$

where

$$\mathscr{A}_r^{\mathrm{an}} = \bar{\boldsymbol{\sigma}}_r - \boldsymbol{X}_r, \quad \boldsymbol{X}_r = \frac{\partial(w^{\mathrm{an}})^{(r)}}{\partial\boldsymbol{\varepsilon}^{\mathrm{an}}}(\boldsymbol{\varepsilon}_r^{\mathrm{an}}), \quad \mathscr{A}_r^{\boldsymbol{\beta}} = -\frac{\partial(w^{\boldsymbol{\beta}})^{(r)}}{\partial\boldsymbol{\beta}}(\boldsymbol{\beta}_r). \tag{31}$$

4. Nonuniform transformation field analysis (NTFA)

The application of the classical TFA to two-phase systems using plastic strains which are uniform on each of the two phases yields very poor predictions of the overall behavior of the composite (Suquet, 1997). Dvorak et al. (1994) have obtained better results by subdividing each phase into several subdomains (at least those undergoing plastic deformation). The need for a finer subdivision of the phases stems from the intrinsic nonuniformity of the plastic strain field which can be highly heterogeneous even within a single material phase. Unfortunately, as the subdivision is refined, the number of internal variables needed in the effective constitutive relations, although finite, increases prohibitively.

The aim of this section is to build the nonuniformity of the plastic strain field in the transformation fields themselves.

4.1. Nonuniform transformation fields

In order to reduce the number of internal variables, we make the approximate assumption that

H3. The field of anelastic strains is a finite combination of modes $\boldsymbol{\mu}^{(k)}$:

$$\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x}) = \sum_{k=1}^{M} \varepsilon_k^{\mathrm{an}} \boldsymbol{\mu}^{(k)}(\boldsymbol{x}). \tag{32}$$

A direct analogy with crystal plasticity can be drawn from (32). $\boldsymbol{\mu}^{(k)}$ corresponds to the kth slip system in the terminology of crystal plasticity while $\boldsymbol{\varepsilon}_k^{\mathrm{an}}$ corresponds to the magnitude of the slip on this system. With this analogy in mind, it is expected that the "slip" on the kth "system" will depend on the "resolved shear stress" on this "system", i.e. on $\boldsymbol{\sigma} : \boldsymbol{\mu}^{(k)}$.

However, unlike crystal plasticity and unlike the classical transformation field analysis, the modes $\boldsymbol{\mu}^{(k)}$ are *nonuniform* (not even piecewise uniform) and can depend on the position $\boldsymbol{x}$. They are meant to capture the salient features of the plastic flow modes. They are determined either analytically or numerically. Their number, M, can be different (larger or smaller) from the number N of phases. For incompressible plasticity (the case here), the modes $\boldsymbol{\mu}^{(k)}$ are traceless tensor fields. In addition, in order for the $\varepsilon_k^{\mathrm{an}}$ to be homogeneous to a plastic strain, a normalization condition is imposed

$$\left\langle \boldsymbol{\mu}_{\mathrm{eq}}^{(k)} \right\rangle = 1. \tag{33}$$

We further assume that the modes have their support entirely contained in a single material phase. This assumption makes it possible to define $\chi^{(k)}, w^{(k)}, \boldsymbol{L}^{(k)}, \psi^{(k)}, \ldots$ for k varying from 1 to M (and not only to N) as the characteristic function, free energy, elastic moduli and force potential of the phase in which the support of the shape function $\boldsymbol{\mu}^{(k)}$ is contained.

4.2. Influence factors for the NTFA

The constitutive relations are conveniently expressed in terms of the following "generalized" stress, back stress, strain and plastic strain:

$$\tau_k = \langle \boldsymbol{\sigma} : \boldsymbol{\mu}^{(k)} \rangle, \quad x_k = \langle \boldsymbol{X} : \boldsymbol{\mu}^{(k)} \rangle, \quad e_k = \langle \boldsymbol{\varepsilon} : \boldsymbol{\mu}^{(k)} \rangle, \quad e_k^{\mathrm{an}} = \langle \boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\mu}^{(k)} \rangle. \tag{34}$$

Under the approximation (32), (21) becomes

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) : \bar{\boldsymbol{\varepsilon}} + \sum_{\ell=1}^{M} (\boldsymbol{D} * \boldsymbol{\mu}^{(\ell)})(\boldsymbol{x}) \varepsilon_\ell^{\mathrm{an}}. \tag{35}$$

Upon multiplication of Eq. (35) by $\boldsymbol{\mu}^{(k)}$ and averaging over V, one obtains

$$e_k = \boldsymbol{a}_k : \bar{\boldsymbol{\varepsilon}} + \sum_{\ell=1}^{M} D_{k\ell}^N \varepsilon_\ell^{\mathrm{an}}, \tag{36}$$

where the second-order tensors $\boldsymbol{a}_k$ and the influence factors $D_{lk\ell}^N$ (N stands for NTFA) are defined as

$$\boldsymbol{a}_k = \langle \boldsymbol{A}^T : \boldsymbol{\mu}^{(k)} \rangle, \quad D_{k\ell}^N = \langle \boldsymbol{\mu}^{(k)} : \boldsymbol{D} * \boldsymbol{\mu}^{(\ell)} \rangle. \tag{37}$$

Then, since the elastic moduli of the phases are isotropic, characterized by a bulk modulus $k^{(r)}$ and a shear modulus $G^{(r)}$, the "resolved shear stress" τ_k are given by

$$\tau_k = 2G^{(k)}(e_k - e_k^{\mathrm{an}}). \tag{38}$$

Remark. Note that the set $\{\varepsilon_k^{\mathrm{an}}\}$, $k = 1, \ldots, M$ can be replaced by the set $\{e_k^{\mathrm{an}}\}$ $k = 1, \ldots, M$ defined in (34), since

$$e_k^{\mathrm{an}} = \sum_{\ell=1}^{M} g_{k\ell} \varepsilon_\ell^{\mathrm{an}}, \quad \text{where} \quad g_{k\ell} = \langle \boldsymbol{\mu}^{(k)} : \boldsymbol{\mu}^{(\ell)} \rangle. \tag{39}$$

The modes are chosen such that the second order tensor $\boldsymbol{g}$ is invertible which permits to invert (39).

4.3. State laws and complementary laws

It remains to choose the state variables at the macroscopic level and to formulate the state laws and the complementary laws. As in the classical TFA, the decomposition of the plastic strain (32) instead of (22) is not sufficient to close the model. The formulation of the evolution equations for the state variables requires an approximation analogous to assumption H2 in Section 3. There are several ways in which this approximation can be made and we will describe two of them.

4.3.1. Uncoupled model

First, we have to specify the macroscopic state variables of the model. At each mode $\boldsymbol{\mu}^{(k)}$ is associated a plastic strain $\varepsilon_k^{\mathrm{an}}$ by means of the decomposition (32).

The field of the internal variables $\boldsymbol{\beta}(\boldsymbol{x})$ is reduced to a set of tensorial variables $\boldsymbol{\beta}_k$, $k = 1, \ldots, M$ associated with each mode. The *reduced* macroscopic state variables of the model are the overall strain $\bar{\boldsymbol{\varepsilon}}$ and the set of all the $\varepsilon_k^{\mathrm{an}}$'s, and $\boldsymbol{\beta}_k$'s, k varying from 1 to M

$$\text{State variables}: \quad (\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}), \quad \tilde{\boldsymbol{\alpha}} = \{\varepsilon_k^{\mathrm{an}}, \boldsymbol{\beta}_k\}_{k=1,\ldots,M}. \tag{40}$$

Part of the state laws are given by (38). To derive the other state equations, we multiply the first equation in (6) by $\boldsymbol{\mu}^{(k)}$ and average over V

$$x_k = \langle \boldsymbol{X} : \boldsymbol{\mu}^{(k)} \rangle = \frac{2}{3} \left\langle \frac{\partial w^{\mathrm{an}}}{\partial e}(\varepsilon_{\mathrm{eq}}^{\mathrm{an}}) \frac{\boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\mu}^{(k)}}{\varepsilon_{\mathrm{eq}}^{\mathrm{an}}} \right\rangle. \tag{41}$$

Since $\boldsymbol{\mu}^{(k)}$ is supported in a single phase, w^{an} can be replaced by $(w^{\mathrm{an}})^{(k)}$ without loss of generality. To further simplify this relation *we replace* $\varepsilon_{\mathrm{eq}}^{\mathrm{an}}$ by $|e_k^{\mathrm{an}}|$ and we get

$$x_k = \frac{2}{3} \frac{\partial (w^{\mathrm{an}})^{(k)}}{\partial e}(e_k^{\mathrm{an}}) \frac{\langle \boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\mu}^{(k)} \rangle}{|e_k^{\mathrm{an}}|} = \frac{2}{3} \frac{\partial (w^{\mathrm{an}})^{(k)}}{\partial e}(e_k^{\mathrm{an}}) \frac{e_k^{\mathrm{an}}}{|e_k^{\mathrm{an}}|}. \tag{42}$$

The other state equations define the forces associated with $\boldsymbol{\beta}_k$

$$\mathscr{A}_k^{\boldsymbol{\beta}} = -\frac{\partial (w^{\boldsymbol{\beta}})^{(k)}}{\partial \boldsymbol{\beta}}(\beta_k). \tag{43}$$

To derive the complementary laws we multiply the second equation in (6) by $\boldsymbol{\mu}^{(k)}$ and average over V

$$\dot{e}_k^{\mathrm{an}} = \langle \dot{\boldsymbol{\varepsilon}}^{\mathrm{an}} : \boldsymbol{\mu}^{(k)} \rangle = \frac{3}{2} \left\langle \frac{\partial \psi}{\partial a}(\mathscr{A}_{\mathrm{eq}}^{\mathrm{an}}, \mathscr{A}^{\beta}) \frac{\mathscr{A}^{\mathrm{an}} : \boldsymbol{\mu}^{(k)}}{\mathscr{A}_{\mathrm{eq}}^{\mathrm{an}}} \right\rangle. \tag{44}$$

Since $\boldsymbol{\mu}^{(k)}$ is supported in a single phase, we can replace ψ by $\psi^{(k)}$. To further simplify this relation *we replace* $\mathscr{A}_{\mathrm{eq}}^{\mathrm{an}}$ by $|\mathscr{A}_k^{\mathrm{an}}|$ and $\mathscr{A}^{\boldsymbol{\beta}}$ by $\mathscr{A}_k^{\boldsymbol{\beta}}$ and we get

$$\dot{e}_k^{\mathrm{an}} = \frac{3}{2} \frac{\partial \psi^{(k)}}{\partial a}(\mathscr{A}_k^{\mathrm{an}}, \mathscr{A}_k^{\boldsymbol{\beta}}) \frac{\mathscr{A}_k^{\mathrm{an}}}{|\mathscr{A}_k^{\mathrm{an}}|}. \tag{45}$$

The evolution equations for the $\boldsymbol{\beta}_k$ are taken in standard form

$$\dot{\boldsymbol{\beta}}_k = \frac{\partial \psi^{(k)}}{\partial \mathscr{A}^{\boldsymbol{\beta}}}(|\mathscr{A}_k^{\mathrm{an}}|, \mathscr{A}_k^{\boldsymbol{\beta}}). \tag{46}$$

In summary, the constitutive relations for the uncoupled model are

$$\left.\begin{aligned}
&\mathscr{A}_k^{\mathrm{an}} = \tau_k - x_k, \quad \tau_k \text{ given by (38)},\\
&x_k = \frac{2}{3}\frac{\partial (w^{\mathrm{an}})^{(k)}}{\partial e}(e_k^{\mathrm{an}})\frac{e_k^{\mathrm{an}}}{|e_k^{\mathrm{an}}|},\\
&\mathscr{A}_k^{\boldsymbol{\beta}} = -\frac{\partial (w^{\boldsymbol{\beta}})^{(k)}}{\partial \boldsymbol{\beta}}(\beta_k),\\
&\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\frac{\partial \psi^{(k)}}{\partial a}(|\mathscr{A}_k^{\mathrm{an}}|, \mathscr{A}_k^{\boldsymbol{\beta}})\frac{\mathscr{A}_k^{\mathrm{an}}}{|\mathscr{A}_k^{\mathrm{an}}|},\\
&\dot{\boldsymbol{\beta}}_k = \frac{\partial \psi^{(k)}}{\partial \mathscr{A}^{\boldsymbol{\beta}}}(|\mathscr{A}_k^{\mathrm{an}}|, \mathscr{A}_k^{\boldsymbol{\beta}}).
\end{aligned}\right\} \tag{47}$$

This system of differential equations is solved along a prescribed path, either in the space of macroscopic stresses or in the space of macroscopic strains. The resolution of the system yields the history of the e_k^{an}'s from which the history of the $\varepsilon_k^{\mathrm{an}}$'s can be obtained by inversion of (39).

Finally, once the internal variables $\varepsilon_k^{\mathrm{an}}$ are determined, the effective constitutive relations for the composite are obtained by averaging the stress field which results from (3), (32) and (35)

$$\bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{L}} : \bar{\boldsymbol{\varepsilon}} + \sum_{k=1}^{M}(\langle \boldsymbol{L} : \boldsymbol{D} * \boldsymbol{\mu}^{(k)}\rangle - \langle \boldsymbol{L} : \boldsymbol{\mu}^{(k)}\rangle)\varepsilon_k^{\mathrm{an}}. \tag{48}$$

The tensors $\langle \boldsymbol{L} : \boldsymbol{D} * \boldsymbol{\mu}^{(k)}\rangle$ and $\langle \boldsymbol{L} : \boldsymbol{\mu}^{(k)}\rangle$ are computed once for all.

4.3.2. Coupled model

Depending on the level of accuracy that one wants to reach, several modes have to be introduced for each individual phase. As a consequence the number of internal variables can be too high. It is possible to reduce further this number by attaching an internal variable $\boldsymbol{\beta}_r$ to each phase and not to each mode and by coupling the different modes supported by the same phase. The resulting coupled model goes as follows.

The state variables are the plastic strains $\varepsilon_k^{\mathrm{an}}$'s on each mode, k varying from 1 to M and the $\boldsymbol{\beta}_r$, r varying from 1 to N

$$\text{State variables}: \quad (\bar{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\alpha}}), \quad \tilde{\boldsymbol{\alpha}} = \{\varepsilon_k^{\mathrm{an}},\}_{k=1,\ldots,M}, \quad \{\boldsymbol{\beta}_r\}_{r=1,\ldots,N}. \tag{49}$$

Then, coming back (41) we note that since $\boldsymbol{\mu}^{(k)}$ is supported by a single phase r, w can be replaced by $w^{(k)}$ without loss of generality. Then $\varepsilon_{\mathrm{eq}}^{\mathrm{an}}$ *is approximated by* $|e_k^{\mathrm{an}}|$ and $\boldsymbol{\beta}$ *is approximated by* $\boldsymbol{\beta}_r$ to get

$$x_k = \frac{2}{3}\frac{\partial (w^{\mathrm{an}})^{(k)}}{\partial e}(e_k^{\mathrm{an}})\frac{e_k^{\mathrm{an}}}{|e_k^{\mathrm{an}}|}. \tag{50}$$

The rest of the state equations permits to define the forces associated with $\boldsymbol{\beta}_r$

$$\mathscr{A}_r^{\boldsymbol{\beta}} = -\frac{\partial (w^{\boldsymbol{\beta}})^{(r)}}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}_r). \tag{51}$$

Then, coming back to (44), ψ is replaced by $\psi^{(k)}$, $\mathscr{A}_{\mathrm{eq}}^{\mathrm{an}}$ *is replaced by*

$$a_r^{\mathrm{an}} = \left(\sum_{k=1}^{M(r)} |\mathscr{A}_k^{\mathrm{an}}|^2\right)^{1/2}, \text{ where } M^{(r)} \text{ is the number of modes in phase } r,$$

and $\mathscr{A}^{\beta}$ *by* $\mathscr{A}_r^{\beta}$. We finally obtain

$$\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\frac{\partial \psi^{(k)}}{\partial a}(a_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}})\frac{\mathscr{A}_k^{\mathrm{an}}}{a_r^{\mathrm{an}}}. \tag{52}$$

The evolution equations for the $\boldsymbol{\beta}_r$ are given in standard form

$$\dot{\boldsymbol{\beta}}_r = \frac{\partial \psi^{(r)}}{\partial \mathscr{A}^{\boldsymbol{\beta}}}(a_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}}). \tag{53}$$

In summary the constitutive relations for the coupled model are

$$\left.\begin{aligned}
&\mathscr{A}_k^{\mathrm{an}} = \tau_k - x_k, \quad \tau_k \text{ given by (38)},\\
&x_k = \frac{2}{3}\frac{\partial (w^{\mathrm{an}})^{(k)}}{\partial e}(e_k^{\mathrm{an}})\frac{e_k^{\mathrm{an}}}{|e_k^{\mathrm{an}}|},\\
&\mathscr{A}_r^{\boldsymbol{\beta}} = -\frac{\partial (w^{\boldsymbol{\beta}})^{(r)}}{\partial \boldsymbol{\beta}}(\beta_r),\\
&\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\frac{\partial \psi^{(k)}}{\partial a}(a_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}})\frac{\mathscr{A}_k^{\mathrm{an}}}{a_r^{\mathrm{an}}}, \quad a_r^{\mathrm{an}} = \left(\sum_{k=1}^{M(r)} |\mathscr{A}_k^{\mathrm{an}}|^2\right)^{1/2},\\
&\dot{\boldsymbol{\beta}}_r = \frac{\partial \psi^{(r)}}{\partial \mathscr{A}^{\boldsymbol{\beta}}}(a_r^{\mathrm{an}}, \mathscr{A}_r^{\boldsymbol{\beta}}),
\end{aligned}\right\} \tag{54}$$

where r is the phase containing the support of $\boldsymbol{\mu}^{(k)}$.

4.4. Two-phase composites

All the examples discussed in Section 5 concern two-phase composites made of linear elastic fibers in an elastoplastic matrix with isotropic or kinematic hardening. The reduced constitutive relations are given in detail for both the uncoupled model and the coupled model in the only nonlinear constituent, the matrix.

Elastoplasticity with isotropic hardening. For isotropic hardening, the internal variables are the plastic strain and $\boldsymbol{\beta}$ reduces to the cumulated plastic strain p. The stored energy is a function $w^{\mathrm{p}}(p)$. The force $\mathscr{A}^{\mathrm{an}}$ reduces to the stress $\boldsymbol{\sigma}$, whereas the force $\mathscr{A}^{\boldsymbol{\beta}}$ reduces to $-R(p)$ defined in (8). The force potential ψ is the limit as n goes to $+\infty$ of the potential (9)

$$\psi(\boldsymbol{\sigma}) = 0, \quad \text{if } \sigma_{\mathrm{eq}} \leqslant R(p), \quad +\infty \quad \text{otherwise}, \tag{55}$$

and the exact complementary law reads

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{an}} = \frac{3}{2}\dot{p}\frac{\boldsymbol{s}}{\sigma_{\mathrm{eq}}}, \quad \dot{p} = 0 \quad \text{if } \sigma_{\mathrm{eq}} < R(p), \quad \dot{p} \geqslant 0 \quad \text{if } \sigma_{\mathrm{eq}} = R(p). \tag{56}$$

The evolution equations for the uncoupled model are, for each mode $k = 1, \ldots, M$

$$\left.\begin{aligned}
&\tau_k = 2G(e_k - e_k^{\mathrm{an}}), \quad |\tau_k| \leqslant R(p_k),\\
&\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\dot{p}_k\frac{\tau_k}{R(p_k)}, \quad \dot{p}_k = \begin{cases} 0 & \text{if } |\tau_k| < R(p_k),\\ \geqslant 0 & \text{otherwise.}\end{cases}
\end{aligned}\right\} \tag{57}$$

The evolution equations for the coupled model are, for each mode $k = 1, \ldots, M$

$$\left.\begin{aligned}
&\tau_k = 2G(e_k - e_k^{\mathrm{an}}), \quad \left(\sum_{k=1}^{M} |\tau_k|^2\right)^{1/2} \leqslant R(p)\\
&\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\dot{p}\frac{\tau_k}{R(p)}, \quad \dot{p} = \begin{cases} 0 & \text{if } \left(\sum_{k=1}^{M} |\tau_k|^2\right)^{1/2} < R(p),\\ \geqslant 0 & \text{otherwise.}\end{cases}
\end{aligned}\right\} \tag{58}$$

In the uncoupled model the M generalized stresses τ_k are subject to M different yield conditions depending on M different plastic multipliers p_k, whereas in the coupled model there is only one yield condition and only one plastic multiplier p.

Elastoplasticity with linear kinematic hardening. In this case, the only internal variable is the plastic strain $\boldsymbol{\varepsilon}^{\mathrm{an}}$ (no $\boldsymbol{\beta}$). The stored energy for the elastoplastic phase reads

$$w^{\mathrm{an}}(\boldsymbol{\varepsilon}^{\mathrm{an}}) = \frac{1}{2} H\boldsymbol{\varepsilon}^{\mathrm{an}} : \boldsymbol{\varepsilon}^{\mathrm{an}}. \tag{59}$$

The anelastic force is $\mathscr{A}^{\mathrm{an}} = \boldsymbol{\sigma} - \boldsymbol{X}$ with $\boldsymbol{X} = H\boldsymbol{\varepsilon}^{\mathrm{an}}$. The force potential ψ is the limit as n goes to $+\infty$ of the potential (9)

$$\psi(\mathscr{A}^{\mathrm{an}}) = 0 \quad \text{if } \mathscr{A}^{\mathrm{an}}_{\mathrm{eq}} - \sigma_0 \leqslant 0, \quad +\infty \quad \text{otherwise}, \tag{60}$$

and the exact complementary law reads

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{an}} = \frac{3}{2}\dot{p}\frac{\mathscr{A}^{\mathrm{an}}}{\mathscr{A}^{\mathrm{an}}_{\mathrm{eq}}}, \quad \dot{p} = 0 \quad \text{if } \mathscr{A}^{\mathrm{an}}_{\mathrm{eq}} - \sigma_0 < 0, \quad \dot{p} \geqslant 0 \quad \text{if } \mathscr{A}^{\mathrm{an}}_{\mathrm{eq}} = \sigma_0. \tag{61}$$

The evolution equations for the uncoupled model are, for each mode $k = 1, \ldots, M$

$$\left.\begin{aligned} &\tau_k = 2G(e_k - e_k^{\mathrm{an}}), \quad x_k = He_k^{\mathrm{an}}, \quad |\tau_k - x_k| \leqslant \sigma_0, \\ &\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\dot{p}_k \frac{\tau_k - x_k}{|\tau_k - x_k|}, \quad \dot{p}_k = \begin{cases} 0 & \text{if } |\tau_k - x_k| < \sigma_0, \\ \geqslant 0 & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{62}$$

The evolution equations for the coupled model are, for each mode $k = 1, \ldots, M$

$$\left.\begin{aligned} &\tau_k = 2G(e_k - e_k^{\mathrm{an}}), \quad x_k = He_k^{\mathrm{an}}, \quad \left(\sum_{k=1}^{M} |\tau_k - x_k|^2\right)^{1/2} \leqslant \sigma_0, \\ &\dot{e}_k^{\mathrm{an}} = \frac{3}{2}\dot{p}\frac{\tau_k - x_k}{\sigma_0}, \quad \dot{p} = \begin{cases} 0 & \text{if } \left(\sum_{k=1}^{M} |\tau_k - x_k|^2\right)^{1/2} < \sigma_0, \\ \geqslant 0 & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{63}$$

As in the case of isotropic hardening the uncoupled model involves M different yield conditions and M different plastic multipliers p_k, whereas the coupled model has only one yield condition and one plastic multiplier p.

5. Examples

The relative merits of the above models, TFA, uncoupled NTFA and coupled NTFA, are assessed by comparison with full numerical simulation of the response of the r.v.e. performed by a method based on fast Fourier transforms (Moulinec and Suquet, 1998; Michel et al., 1999). All the examples presented in this section pertain to two-phase composites comprised of linear elastic fibers with identical circular cross section embedded in a nonlinear elastoplastic and hardening matrix.

5.1. Configurations and material data

Three typical arrangements of the fibers are considered (Fig. 1), two hexagonal configurations with fiber volume fractions $c_{\mathrm{f}} = 0.25$ and $c_{\mathrm{f}} = 0.50$ and a configuration where 64 fibers are arranged randomly (subject to periodicity conditions and impenetrability conditions) in a square cell with volume fraction $c_{\mathrm{f}} = 0.25$.

Three typical hardening laws are used in the analysis, linear isotropic hardening, nonlinear isotropic hardening and linear kinematic hardening. The material data are different for each case

Matrix with linear isotropic hardening. Matrix yield stress: $\sigma_{\mathrm{eq}} \leqslant \sigma_0 + hp$.

$$\left.\begin{aligned} &\text{Fibers}: \quad E_{\mathrm{f}} = 414 \text{ GPa}, \quad \nu_{\mathrm{f}} = 0.19, \\ &\text{Matrix}: \quad E_{\mathrm{m}} = 99.5 \text{ GPa}, \quad \nu_{\mathrm{m}} = 0.3, \quad \sigma_0 = 510 \text{ MPa}, \quad h = 5100 \text{ MPa}. \end{aligned}\right\} \tag{64}$$

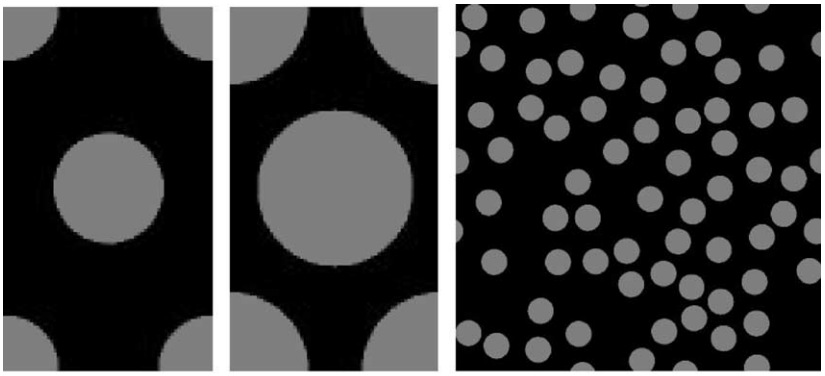

Fig. 1. Microstructures used in the comparison between the reduced models and exact computations: (a) Hexagonal cell, volume fraction of fibers = 0.25 (discretization 125×217 pixels); (b) hexagonal cell, volume fraction of fibers = 0.5 (discretization 125×217 pixels); (c) random arrangement of fibers, volume fraction of fibers = 0.25 (discretization 1215×1215 pixels).

Matrix with nonlinear isotropic hardening. Matrix yield stress: $\sigma_{\text{eq}} \leqslant \sigma_0 + hp^m$.

$$\left.\begin{array}{l}\text{Fibers}: \quad E_{\text{f}} = 400\ \text{GPa}, \quad \nu_{\text{f}} = 0.2 \\ \text{Matrix}: \quad E_{\text{m}} = 75\ \text{GPa}, \quad \nu_{\text{m}} = 0.3\ \text{GPa}, \quad \sigma_0 = 75\ \text{MPa}, \quad h = 416.5\ \text{MPa}, \quad m = 0.3895.\end{array}\right\} \tag{65}$$

Matrix with linear kinematic hardening. Matrix yield stress: $(\sigma - X)_{\text{eq}} \leqslant \sigma_0, \quad \boldsymbol{X} = H\boldsymbol{\varepsilon}^{\text{an}}$.

$$\left.\begin{array}{l}\text{Fibers}: \quad E_{\text{f}} = 414\ \text{GPa}, \quad \nu_{\text{f}} = 0.19, \\ \text{Matrix}: \quad E_{\text{m}} = 99.5\ \text{GPa}, \quad \nu_{\text{m}} = 0.3, \quad \sigma_0 = 510\ \text{MPa}, \quad H = 2400\ \text{MPa}.\end{array}\right\} \tag{66}$$

5.2. Loading

The macroscopic stress applied to the r.v.e is imposed in a fixed direction in stress space

$$\bar{\boldsymbol{\sigma}} = \bar{\sigma}(t)\boldsymbol{\Sigma}^0. \tag{67}$$

The loading is applied by increasing incrementally the "control parameter" $\bar{\boldsymbol{\varepsilon}} : \boldsymbol{\Sigma}^0$. In other words, the direction of overall stress is imposed, the magnitude of the overall strain in this direction is the control parameter, the magnitude of the stress $\bar{\sigma}(t)$ and the components of the overall strain in the directions perpendicular to $\boldsymbol{\Sigma}^0$ are outputs of the computation. The control parameter can be increased monotonically (in which case it plays the role of an artificial time) or cycled. The results obtained for the stress–strain relations are plotted in a diagram $(\bar{\sigma}, \bar{\boldsymbol{\varepsilon}} : \boldsymbol{\Sigma}^0)$.

In most of the examples shown here, four different stress states were considered

$$\boldsymbol{\Sigma}^{(1)} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \quad \boldsymbol{\Sigma}^{(2)} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1, \quad \boldsymbol{\Sigma}^{(3)} = \boldsymbol{\Sigma}^{(1)} + \frac{1}{2}\boldsymbol{\Sigma}^{(2)}, \quad \boldsymbol{\Sigma}^{(4)} = \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(2)}. \tag{68}$$

5.3. Plastic modes

The modes $\boldsymbol{\mu}^{(k)}$ have to be specified. Their choice is left to the user and is a keypoint in the accuracy of the method. In the examples presented below they were chosen to be *actual plastic strain fields* in the composite under certain loadings. More specifically, the modes were determined numerically by simulating the response of the composite along monotone loading paths in the space of macroscopic stresses corre-

sponding to uniaxial tension and pure shear, respectively, in (67), i.e. $\boldsymbol{\Sigma}^0$ being either $\boldsymbol{\Sigma}^{(1)}$ or $\boldsymbol{\Sigma}^{(2)}$. The response of the composite was computed "exactly" (up to numerical accuracy). The plastic modes $\boldsymbol{\mu}^{(k)}$, $k = 1, 2$ were taken to be the plastic strain fields $\boldsymbol{\varepsilon}^{\mathrm{an}}(\boldsymbol{x})$ when $\bar{\boldsymbol{\varepsilon}} : \boldsymbol{\Sigma}^{(k)} = 5\%$. This relatively high value of the strain corresponds to a fully developed plastic strain field. More modes could have been added to the analysis (at the expense of additional internal variables) by taking plastic strain fields at intermediate values of $\bar{\boldsymbol{\varepsilon}} : \boldsymbol{\Sigma}^{(k)}$.

For simplicity, we restricted ourself to two modes only (the resulting constitutive relations for the composite can therefore be expressed in terms of only two internal scalar variables) in each of the nine cases investigated (three different configurations, three different behaviors). The support of the modes were contained in the matrix only. Each mode has four independent components, three in-plane components and one component in the direction perpendicular to the plane. The four different components of the mode $\boldsymbol{\mu}^{(1)}$ for the hexagonal array at a fiber volume fraction of 0.25 are shown in Fig. 2. In Fig. 3 two different components of the modes $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$ for the random configuration are shown. As can be seen from these figures the modes are fairly inhomogeneous. In our case the elastoplastic problem was solved using a method based on fast Fourier transforms in generalized plane strains (Michel et al., 1999), but any other computational method could have been used.

It results from this construction that the modes depend on the microstructure of the composite and on the behavior of the matrix.

5.4. Discussion of the results

Monotone and cyclic loadings were considered.

Monotone loadings. The predictions of the different models (TFA, uncoupled NTFA, coupled NTFA) are shown in Figs. 4–6 obtained with the following data. Fig. 4 corresponds to the hexagonal cell with 25% of fibers and a matrix with linear isotropic hardening. Fig. 5 corresponds to the random configuration with 25% of fibers and a matrix with nonlinear isotropic hardening. Fig. 6 corresponds to the hexagonal cell with 50% of fibers and a matrix with nonlinear isotropic hardening. The TFA was implemented with two subdomains, the matrix and the fibers (called the "two-point" averaging approach in Fish et al., 1997), and one tensorial internal variable (uniform plastic strain in the matrix) with three independent components. The uncoupled NTFA and of the coupled NTA were implemented with two modes in the matrix (two scalar internal variables). The result of an "exact" computation (up to numerical errors) performed by a computational method based on FFT (Moulinec and Suquet, 1998) is also shown on these figures.

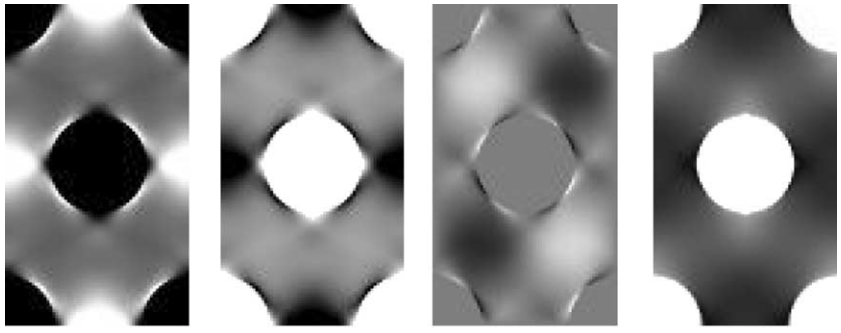

Fig. 2. Hexagonal cell, $c_f = 0.25$. Matrix with nonlinear isotropic hardening. The four components of the mode $\boldsymbol{\mu}^{(1)}$ (corresponding to uniaxial tension) are shown: (a) $\mu_{11}^{(1)}$; (b) $\mu_{22}^{(1)}$; (c) $\mu_{12}^{(1)}$ and (d) $\mu_{33}^{(1)}$.

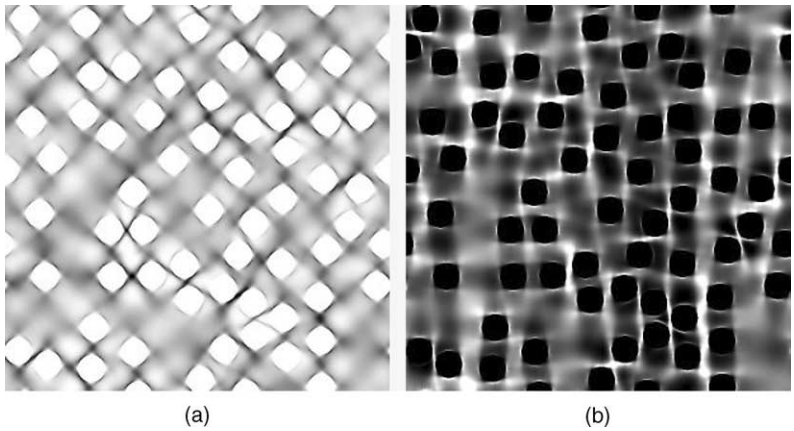

Fig. 3. Random configuration. Matrix with nonlinear isotropic hardening. Two components of the modes $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$ are shown: (a) $\mu_{22}^{(1)}$ and (b) $\mu_{12}^{(2)}$.

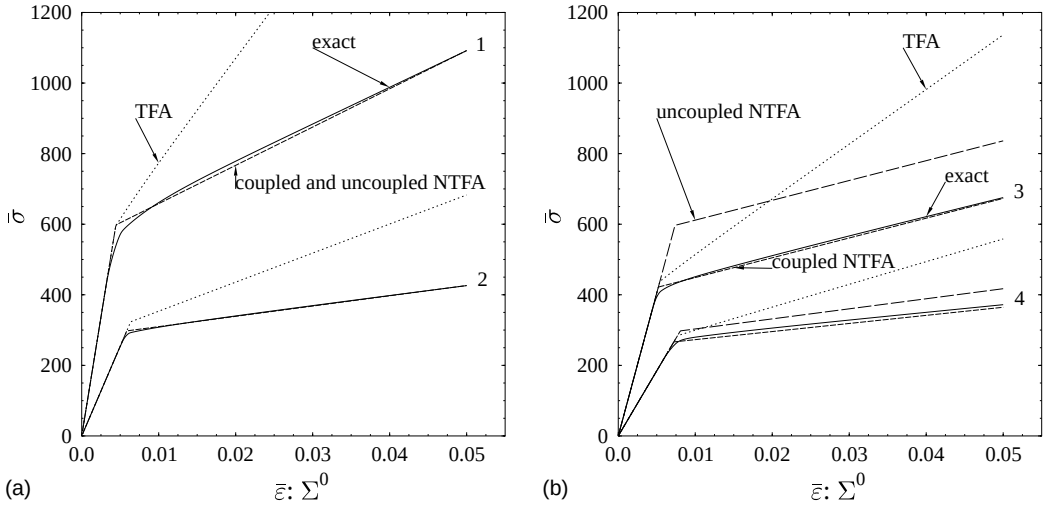

Fig. 4. Hexagonal cell, $c_f = 0.25$. Isotropic nonlinear hardening. Predictions of the TFA (dotted lines), uncoupled NTFA (long dash), coupled NTFA (short dashed) and exact results (solid lines): (a) loading along $\boldsymbol{\Sigma}^{(1)}$ and $\boldsymbol{\Sigma}^{(2)}$ and (b) loading along $\boldsymbol{\Sigma}^{(3)}$ and $\boldsymbol{\Sigma}^{(4)}$ (see (67) and (68) for the definition of the loading).

The predictions of the TFA in its original form (only two domains) are seen to be unrealistic for all configurations and all matrix behaviors. This was expected from other results presented in the literature, as already mentioned.

The curves on the left hand side of the figures correspond precisely to the loadings $\boldsymbol{\Sigma}^{(1)}$ and $\boldsymbol{\Sigma}^{(2)}$ from which the modes $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$ were taken. The agreement with the exact result is very good, not only at the strain where the modes were extracted (5%), but also in the whole range of strains considered. The curves on the right-hand side of the figures correspond to the loadings $\boldsymbol{\Sigma}^{(3)}$ and $\boldsymbol{\Sigma}^{(4)}$ which were not used in the determination of the plastic modes. It is seen that the coupled theory is in better agreement with the exact results than the uncoupled theory.

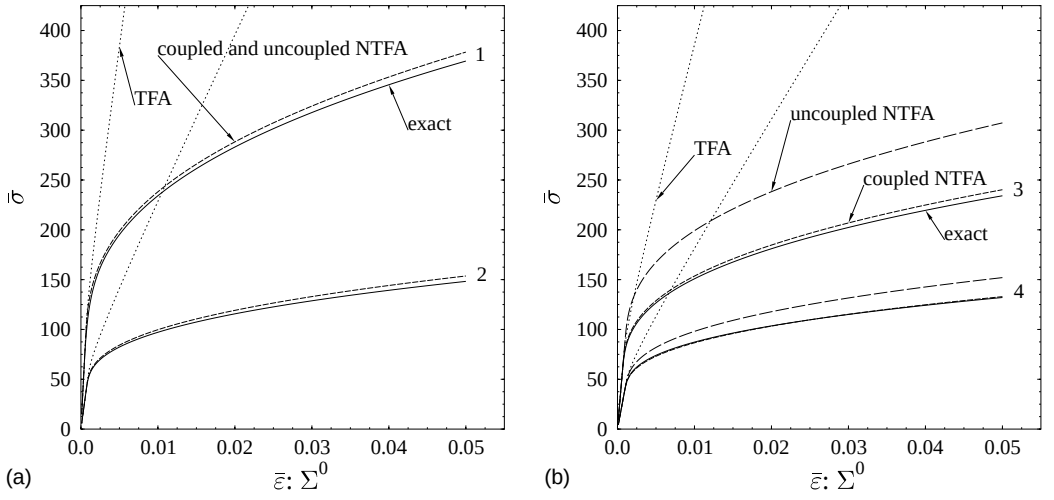

Fig. 5. Hexagonal cell, $c_f = 0.5$. Isotropic nonlinear hardening. Predictions of the TFA (dotted lines), uncoupled NTFA (long dash), coupled NTFA (short dashed) and exact results (solid lines): (a) loading along $\mathbf{\Sigma}^{(1)}$ and $\mathbf{\Sigma}^{(2)}$ and (b) loading along $\mathbf{\Sigma}^{(3)}$ and $\mathbf{\Sigma}^{(4)}$ (see (67) and (68) for the definition of the loading).

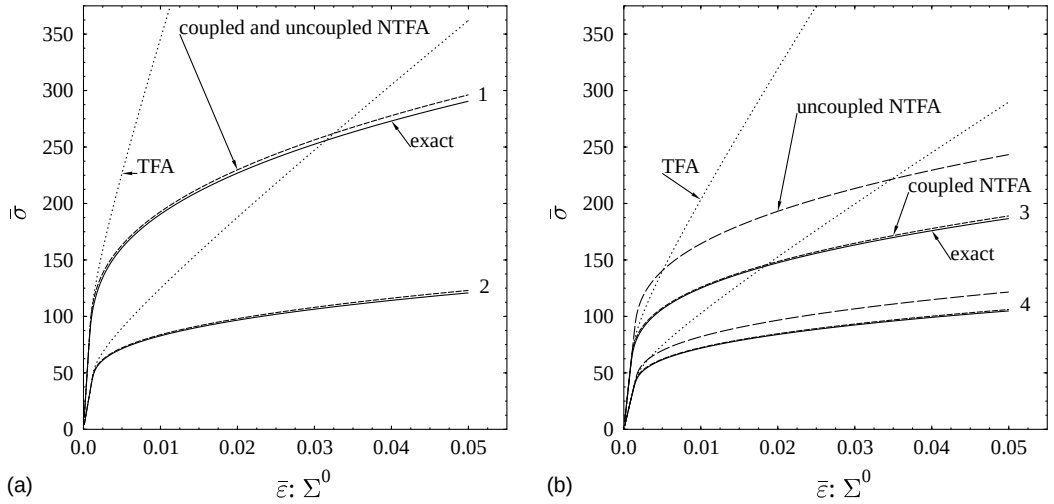

Fig. 6. Random configuration. Isotropic nonlinear hardening. Predictions of the TFA (dotted lines), uncoupled NTFA (long dash), coupled NTFA (short dashed) and exact results (solid lines): (a) Loading along $\mathbf{\Sigma}^{(1)}$ and $\mathbf{\Sigma}^{(2)}$ and (b) loading along $\mathbf{\Sigma}^{(3)}$ and $\mathbf{\Sigma}^{(4)}$.

Cyclic loadings. Finally the predictions of the coupled model (the most successful under monotone loading) are shown in Fig. 7 for the random configuration for a matrix with isotropic hardening (left) and kinematic hardening (right). Both parts of figures show a quite satisfactory agreement between the model and the full simulations, except in the reloading portions of the curves. The exact curve is smoother, corresponding to a stronger overall kinematic hardening, which, however, affects only a small part of the response. It is our expectation (but remains to be checked) that the agreement could be improved by increasing the number of modes considered in the analysis, rather than by changing the modes.

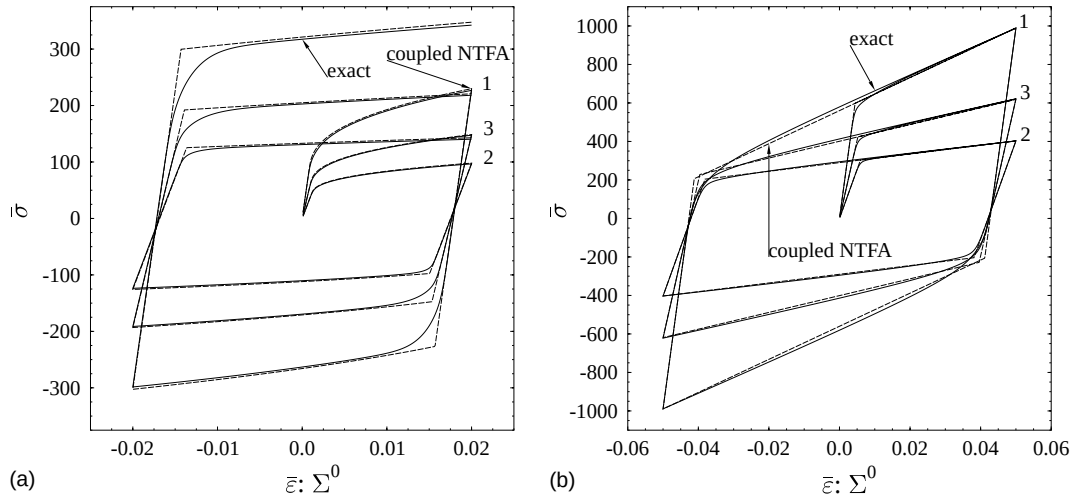

Fig. 7. Random configuration. Cyclic loading. Loading along $\mathbf{\Sigma}^{(1)}$, $\mathbf{\Sigma}^{(2)}$ and $\mathbf{\Sigma}^{(3)}$. Predictions of the coupled NTFA (dashed lines) and exact results (solid lines): (a) matrix with isotropic nonlinear hardening and (b) matrix with kinematic linear hardening.

6. Conclusion

An approximate model for describing the overall hardening of elastoplastic or elastoviscoplastic composites has been described. The main feature of the model is that it makes use of *nonuniform* transformation fields, generalizing the idea of Dvorak (1992). The implementation of the method consists in two different steps

- First, plastic modes representative of the plastic strain fields expected at the microscopic level, have to be determined. This can be done numerically by simulating the plastic fields which develop at the microscopic level under well-chosen loadings.
- Second, constitutive equations for the generalized components of the plastic strains (for each plastic mode) have to be derived. This can be done in uncoupled or coupled form. Comparisons with numerical simulations show that the coupled model is more accurate.

The advantages of the method is that, if the modes are suitably chosen, the number of internal variables of the model is kept low whereas the prediction is accurate. On the other hand, the modes have to be precomputed and their choice depend on the problem at hand (configuration, loading cases).

References

Bui, H., 1970. Evolution de la frontière du domaine élastique des métaux avec l'écrouissage plastique et le comportement élastoplastique d'un agrégat de cristaux cubiques. Mémor. l'Artill. Franç. 1, 141–165.

Chaboche, J., Kruch, S., Maire, J., Pottier, T., 2001. Towards a micromechanics based inelastic and damage modeling of composites. Int. J. Plast. 17, 411–439.

Dvorak, G., 1992. Transformation field analysis of inelastic composite materials. Proc. R. Soc. London A 437, 311–327.

Dvorak, G., Bahei-El-Din, Y., 1987. A bimodal plasticity theory of fibrous composite materials. Acta Mech. 69, 219–241.

Dvorak, G., Benveniste, Y., 1992. On transformation strains and uniform fields in multiphase elastic media. Proc. R. Soc. London A 437, 291–310.

Dvorak, G., Rao, M., 1976. Axisymmetric plasticity theory of fibrous composites. Int. J. Eng. Sci. 14, 361–373.

Dvorak, G., Bahei-El-Din, Y., Macheret, Y., Liu, C., 1988. An experimental study of elastic-plastic behavior of a fibrous Boron–Aluminum composite. J. Mech. Phys. Solids 36, 655–687.

Dvorak, G., Bahei-El-Din, Y., Wafa, A., 1994. The modeling of inelastic composite materials with the transformation field analysis. Model. Simulat. Mater. Sci. Eng. 2, 571–586.

Fish, J., Yu, Q., 2002. Computational Mechanics of Fatigue and Life Predictions for Composite Materials and Structures. Comput. Methods Appl. Mech. Eng. 191, 4827–4849.

Fish, J., Shek, K., Pandheeradi, M., Shepard, M., 1997. Computational plasticity for composite structures based on mathematical homogenization: theory and practice. Comput. Methods Appl. Mech. Eng. 148, 53–73.

Germain, P., Nguyen, Q., Suquet, P., 1983. Continuum thermodynamics. J. Appl. Mech. 50, 1010–1020.

Halphen, B., Nguyen, Q., 1975. Sur les matériaux standard généralisés. J. Mécanique 14, 39–63.

Kattan, P., Voyiadjis, G., 1993. Overall damage and elastoplastic deformation in fibrous metal matrix composites. Int. J. Plast. 9, 931–949.

Lemaitre, J., Chaboche, J., 1988. Mécanique des matériaux solides. Dunod, Paris.

Lubliner, J., 1990. Plasticity Theory. Macmillan, New York.

Mandel, J., 1972. Plasticité classique et Viscoplasticité. In: CISM Lecture Notes, vol. 97. Springer-Verlag, Wien.

Maugin, G., 1992. The Thermomechanics of Plasticity and Fracture. Cambridge University Press, Cambridge.

Michel, J.C., Moulinec, H., Suquet, P., 1999. Effective properties of composite materials with periodic microstructure: a computational approach. Comp. Methods Appl. Mech. Eng. 172, 109–143.

Michel, J.C., Galvanetto, U., Suquet, P., 2000. Constitutive relations involving internal variables based on a micromechanical analysis. In: Maugin, G., Drouot, R., Sidoroff, F. (Eds.), Continuum Thermomechanics: The Art and Science of Modelling Material Behaviour. Klüwer Academic Publishers, Dordrecht, pp. 301–312.

Moulinec, H., Suquet, P., 1998. A numerical method for computing the overall response of nonlinear composites with complex microstructure. Comp. Methods Appl. Mech. Eng. 157, 69–94.

Olson, T., 1994. Improvements on Taylor's upper bound for rigid-plastic bodies. Mater. Sci. Eng. A 175, 15–20.

Ponte Castañeda, P., 1991. The effective mechanical properties of nonlinear isotropic composites. J. Mech. Phys. Solids 39, 45–71.

Ponte Castañeda, P., 1992. New variational principles in plasticity and their application to composite materials. J. Mech. Phys. Solids 40, 1757–1788.

Ponte Castañeda, P., 1996. A second-order theory for nonlinear composite materials. C.R. Acad. Sci. Paris 322 (Sér. II b), 3–10.

Ponte Castañeda, P., Suquet, P., 1997. Nonlinear composites. In: der Giessen, E.V., Wu, T. (Eds.), Advances in Applied Mechanics, vol. 34. Academic Press, New York, pp. 171–302.

Rice, J., 1970. On the structure of stress–strain relations for time-dependent plastic deformation in metals. J. Appl. Mech. 37, 728–737.

Suquet, P., 1985. Local and global aspects in the mathematical theory of Plasticity. In: Sawczuk, A., Bianchi, G. (Eds.), Plasticity Today: Modelling, Methods and Applications. Elsevier, London, pp. 279–310.

Suquet, P., 1987. Elements of homogenization for inelastic solid mechanics. In: Sanchez-Palencia, E., Zaoui, A. (Eds.), Homogenization Techniques for Composite Media. In: Lecture Notes in Physics, vol. 272. Springer Verlag, New York, pp. 193–278.

Suquet, P., 1992. On bounds for the overall potential of power law materials containing voids with arbitrary shape. Mech. Res. Commun. 19, 51–58.

Suquet, P., 1993. Overall potentials and extremal surfaces of power law or ideally plastic materials. J. Mech. Phys. Solids 41, 981–1002.

Suquet, P., 1997. Effective properties of nonlinear composites. In: Suquet, P. (Ed.), Continuum Micromechanics. In: CISM Lecture Notes, vol. 377. Springer Verlag, New York, pp. 197–264.

Talbot, D., Willis, J., 1985. Variational principles for inhomogeneous nonlinear media. IMA J. Appl. Math. 35, 39–54.

Teply, J., Dvorak, G., 1988. Bounds on overall instantaneous properties of elastic-plastic composites. J. Mech. Phys. Solids 36, 29–58.

Willis, J., 1991. On methods for bounding the overall properties of nonlinear composites. J. Mech. Phys. Solids 39, 73–86.